References:

Keller - Introduction to A_∞-Algebras and Modules

深谷賢治 - Deformation Theory, Homological Algebra, and Mirror Symmetry.

① Differential Graded Algebras
② A_∞-Algebras via Higher Products
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⑥ From Algebras to Categories. → 1^{st} July ?

parallel theory

DGLAs

L_∞-Algebras

① Differential Graded Algebras

Definition. (A, d) is a differential graded algebra (DGA) over a commutative ring k, if:

① A is $\mathbb{Z}$-graded: $A = \bigoplus_{n \in \mathbb{Z}} A^n$

② A is an associative unital algebra over k

Product $\mu : A \otimes_k A \to A$ k-bilinear, degree 0

(Graded commutativity: $ab = (-1)^{|a| \cdot |b|} ba$)

③ Differential $d : A \to A$ is k-linear, degree $+1$, and $d^2 = 0$.

④ Graded Leibniz rule: $d(ab) = (da)b + (-1)^{|a|} a(db)$

From DGA we can compute the cohomology $H^n(A) = \dfrac{\ker d^n}{\operatorname{im} d^{n-1}}$.

Similarly $(\mathfrak{g}, d)$ is a differential graded Lie algebra (DGLA) over a commutative ring k, if:

① $\sim$

② $\mathfrak{g}$ is a Lie algebra over k

Graded skew-symmetry: $[a,b] = -(-1)^{|a|\cdot|b|}[b,a]$

Graded Jacobi identity:

$(-1)^{|a|\cdot|b|}[[a,b],c] + (-1)^{|b||c|}[[b,c],a] + (-1)^{|c||a|}[[c,a],b] = 0$

③ $\sim$

④ $\sim$: $d[a,b] = [da, b] + (-1)^{|a|}[a, db]$.

$$\text{DGA} \xrightarrow{\text{associative up to homotopy}} A_\infty\text{-Algebras}$$

$$\text{DGLA} \xrightarrow{\text{Jacobi identity up to homotopy}} L_\infty\text{-Algebras}$$

infinitely many higher products/brackets

② A_∞-Algebras via Higher Products.

Definition (via higher products)

An A_∞-algebra A is a graded k-module equipped with the n-linear products of degree $2-n$:

$$m_n : A^{\otimes n} \to A$$

satisfying:

$$[M_n] \quad \sum_{\substack{r+s+t=n,\\ r,t\geq 0,\ s>0}} (-1)^{r+st}\, m_{r+t+1}(id_A^{\otimes r} \otimes m_s \otimes id_A^{\otimes t}) = 0$$

Note: Koszul sign convention:

$$(f\otimes g)(a\otimes b) = (-1)^{|g||a|} f(a) \otimes g(b)$$

- $[M_1]$: $\mu_1 \circ \mu_1 = 0$
- $[M_2]$: $\mu_1 \circ \mu_2 = \mu_2(\mu_1 \otimes id + id \otimes \mu_1)$
- $[M_3]$: $\mu_2(id \otimes \mu_2 - \mu_2 \otimes id) = \mu_1 \circ \mu_3$

$$+\mu_3(\mu_1 \otimes id \otimes id + id \otimes \mu_1 \otimes id + id \otimes id \otimes \mu_1)$$

If $\mu_n = 0$ for $n \geq 3$, then (A, μ_1, μ_2) is a DGA.

$[M_3]$ is the associativity of μ_2 up to higher products (i.e. μ_3)

Let A, B be A_∞-algebras. An A_∞-morphism $f : A \dashrightarrow B$ is a collection of morphisms $f_n : A^{\otimes n} \to B$ (of degree $1-n$) such that

$$[F_n] \quad \sum_{\substack{r+s+t=n,\\ r,t \geq 0,\, s>0}} (-1)^{r+st} f_{r+t+1}(id_A^{\otimes r} \otimes m_s^A \otimes id_A^{\otimes t}) = \sum_{r=1}^{n} \sum_{i_1+\cdots+i_r=n} (-1)^{\star} m_r^B(f_{i_1} \otimes \cdots \otimes f_{i_r})$$

$$\star = \sum_{k=1}^{r-1} (r-k)(i_k - 1)$$

$f : A \to B$ strict if $f_n = 0$ for $n > 1$.

A strict A_∞-morphism between two DGAs is a DGA morphism:

- $[F_1]$: $f_1 \circ m_1^A = m_1^B \circ f_1$
- $[F_2]$: $f_1 \circ m_2^A - m_2^B(f_1 \otimes f_1) = m_1^B \circ f_2 + f_2(m_1^A \otimes id_A + id_A \otimes m_1^A)$

Composition of A_∞-morphisms $f : B \dashrightarrow C$ and $g : A \dashrightarrow B$

$$(f \circ g)_n = \sum_{r=1}^{n} \sum_{i_1+\cdots+i_r=n} (-1)^{\star} f_r(g_{i_1} \otimes \cdots \otimes g_{i_r})$$

$f : A \dashrightarrow B$ is an A_∞-quasi-isomorphism if $f_1 : A \to B$ is a quasi-isomorphism, i.e. $H^\bullet(f_1) : H^\bullet(A) \to H^\bullet(B)$ is an isomorphism.

③ A_∞-Algebras via Coalgebras

Coalgebras :

Definition 2.36. A **co-associative coalgebra** C over a field k is a k-vector space equipped with a k-linear co-multiplication $\Delta\colon C \to C \otimes_k C$ satisfying the co-associativity:

$$(\mathrm{id}_C \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}_C) \circ \Delta.$$

A co-unit of C is a k-linear map $e\colon C \to k$ satisfying

$$(\mathrm{id}_C \otimes e) \circ \Delta = (e \otimes \mathrm{id}_C) \circ \Delta = \mathrm{id}_C\,.$$

The commutative diagrams are shown below.

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes_k C \arrow[d, "\mathrm{id}_C \otimes \Delta"] \\ C \otimes_k C \arrow[r, "\Delta \otimes \mathrm{id}_C"] & C \otimes_k C \otimes_k C \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes_k C \arrow[d, "\mathrm{id}_C \otimes e"] \\ C \otimes_k C \arrow[r, "e \otimes \mathrm{id}_C"] & C \end{tikzcd}$$

A **graded coalgebra** $C = \bigoplus_{i \in \mathbb{Z}} C^i$ is both a graded vector space and a coalgebra, such that the co-multiplication is compatible with the grading:

$$\Delta(C^i) \subseteq \bigoplus_{p+q=i} (C^p \otimes_k C^q).$$

A morphism $F\colon (C, \Delta) \to (C', \Delta')$ of graded coalgebras is a linear map $F\colon C \to C'$ of degree 0 such that $\Delta' \circ F = (F \otimes F) \circ \Delta$.

Reduced tensor algebra :

$$\otimes^+(V) := \bigoplus_{n=1}^{\infty} V^{\otimes n}, \quad \text{graded by } |x_1 \otimes \cdots \otimes x_n| = \sum_{i=1}^{n} |x_i|$$

(If use reduced symmetric algebra

$$\mathrm{Sym}^+(V) := \bigoplus_{n=1}^{\infty} V^{\otimes n} / \langle x \otimes y - (-1)^{|x||y|} y \otimes x \rangle$$

instead, same construction will get L_∞-algebra)

("bar construction")

Co-multiplication :

$$\Delta(\underbrace{x_1 \otimes \cdots \otimes x_n}_{\otimes^+(V)}) = \sum_{i=1}^{n} (x_1 \otimes \cdots \otimes x_i) \,\underbrace{\bar{\otimes}}_{\otimes}\, (x_{i+1} \otimes \cdots \otimes x_n)$$

"internal" $\otimes^+(V) \otimes \otimes^+(V)$ "external"

$\otimes^+(V)$ is the co-free co-nilpotent coalgebra without co-unit co-generated by V.

Universal property:
$$\begin{tikzcd} & \bigotimes^+(V) \arrow[d] \\ A \arrow[ur, dashed, "\exists!"] \arrow[r, "\forall"] & V \end{tikzcd}$$

Definition 2.39. A **co-derivation** δ of degree d on a graded coalgebra C is a graded k-linear map $\delta\colon C^i \to C^{i+d}$ satisfying the co-Leibniz rule:

$$\Delta \circ \delta = (\delta \otimes \mathrm{id}_C + \mathrm{id}_C \otimes \delta) \circ \Delta$$

Definition (via coalgebra):

An A_∞-algebra (A, μ) is a graded k-module A with a co-derivation μ of degree 1 on the reduced tensor algebra $\bigotimes^+(A[1])$ such that $\mu^2 = 0$. $\Rightarrow$ gets $[M_n]$ after expansion

The "Taylor coefficients" $\mu_n : (A[1])^{\otimes n} \longrightarrow A[1]$ uniquely determines μ by universal property of $\bigotimes^+(A[1])$. They gives the higher products $m_n : A^{\otimes n} \longrightarrow A$ (of deg $2-n$)

Let A, B be A_∞-algebras. An A_∞-morphism $f : A \dashrightarrow B$ is a morphism of graded coalgebras:
$f : \bigotimes^+(A[1]) \to \bigotimes^+(B[1])$ of degree 0 such that $f \circ \mu^A = \mu^B \circ f$. $\Longrightarrow$ gets $[F_n]$ after expansion
The "Taylor coefficients" $\phi_n : A[1]^{\otimes n} \longrightarrow B[1]$ determines f uniquely.

④ Minimal Models

(Everything in this part also works for L_∞-algebras.)

Let A be an A_∞-algebra. We say that A is

1) minimal, if $m_1 = 0$;
2) linear contractible, if $m_n = 0$ for $n \geq 2$ and $H^\bullet(A, m_1) = 0$

0) Observe that an A_∞-quasi-isomorphism between minimal A_∞-algebras is in fact an isomorphism!

i) Homological perturbation lemma.

Let (A, μ_1^A), (B, μ_1^B) be cochain complexes. Let $f_1 : A \to B$ be a quasi-isomorphism with quasi-inverse $\tilde{f}_1 : B \to A$. If μ^B is an A_∞-structure on B extending μ_1^B, then we can construct A_∞-structure μ^A on A extending μ_1^A and A_∞-quasi-isomorphism $f : A \dashrightarrow B$ extending f_1.

ii) Decomposition Theorem.

Every A_∞-algebra is a direct sum of a minimal $\sim$ and a linear contractible $\sim$.

iii) Minimal Model Theorem

Every A_∞-algebra is quasi-isomorphic to a unique (up to A_∞-isomorphism) minimal A_∞-algebra, called the minimal model.

0) + iii) = iv) Every A_∞-quasi-isomorphism admits a quasi-inverse. (What does it mean for the derived cat?)

Starting point of the proof : Hodge decomposition.

Proof. (*Adapted from [AMM02] and [Jur19].*) The first step of the proof is a general fact in linear algebra that any cochain complex of vector spaces is a direct sum of a complex with zero differential and a complex with zero cohomology. For this we consider a cochain complex $(C^\bullet, \mathrm{d})$ of vector spaces. Note that the two short exact sequences

$$0 \longrightarrow \ker \mathrm{d}^n \longrightarrow C^n \longrightarrow \operatorname{im} \mathrm{d}^n \longrightarrow 0$$
$$0 \longrightarrow \operatorname{im} \mathrm{d}^{n-1} \longrightarrow \ker \mathrm{d}^n \longrightarrow \mathrm{H}^n(C^\bullet) \longrightarrow 0$$

split. Therefore we have a decomposition $C^n = Z^n \oplus Z_c^n = B^n \oplus H^n \oplus Z_c^n$, where $Z^n = \ker \mathrm{d}^n$, $B^n = \operatorname{im} \mathrm{d}^{n-1} \cong Z_c^{n-1}$, and $H^n \cong \mathrm{H}^n(C^\bullet)$. We define a linear map $h^n \colon C^n \to C^{n-1}$ by the composition:

$$C^n \twoheadrightarrow B^n \xrightarrow{\sim} Z_c^{n-1} \hookrightarrow C^{n-1}\ .$$

h^n is called the *splitting map*. It follows that $B^n = \operatorname{im}(\mathrm{d}^{n-1} \circ h^n)$ and $Z_c^n = \operatorname{im}(h^{n+1} \circ \mathrm{d}^n)$. Therefore we have a decomposition of the identity map on C^n:

$$\mathrm{id} = p^n + \mathrm{d}^{n-1} \circ h^n + h^{n+1} \circ \mathrm{d}^n,$$

where $p^n \colon C^n \to H^n$ is the projection map. This shows that h is a chain homotopy between id and p. Therefore the cohomology of $H^\bullet \cong \mathrm{H}^\bullet(C^\bullet)$ induced by the projection p is trivial. On the other hand, the projection $1 - p^n \colon C^n \to B^n \oplus Z_c^n$ is chain-homotopic to the zero map. Hence the induced differential on $B^\bullet \oplus Z_c^\bullet$ is zero.

For the rest of the proof, see Fukaya, Bocklandt or my dissertation ...

Let $A \underset{g}{\overset{f}{\dashrightarrow}} B$ be two A_∞-morphisms. We say that $h : A \dashrightarrow B$ is a homotopy from f to g, if $h : \otimes^+ (A[1]) \to \otimes^+ (B[1])$ is a morphism of graded coalgebras of degree -1 such that

$$\Delta h = f \otimes h + h \otimes g, \quad f - g = \mu^B \circ h + h \circ \mu^A$$

Thm. $f : A \dashrightarrow B$ is a A_∞-quasi-isomorphism iff f is a homotopy equivalence.

⑤ A_∞-Modules

Let A be an A_∞-algebra over k. (right A_∞-module)

Definition. An A_∞-module M over A is a graded k-module equipped with the maps of degree $2-n$

$$m_n^M : M \otimes_k A^{\otimes n-1} \to M$$

satisfying :: $$\sum_{\substack{r+s+t=n, \\ r,t\geq 0,\ s>0}} (-1)^{r+st}\, m^M_{r+t+1}\big(\underbrace{id^{\otimes r} \otimes m_s}_{\text{use } m^M_s \text{ instead when } r=0} \otimes id^{\otimes t}\big) = 0$$

A morphism $f : L \dashrightarrow M$ of A_∞-modules over A is a collection of maps of degree $1-n$

$$f_n : L \otimes A^{\otimes n-1} \to M$$

satisfying the similar constraints as $[F_n]$.

Let A be an ordinary unital algebra. Denote by $C_\infty A$ the category of A_∞-modules over A. Then $\mathrm{Mod}\text{-}A$ is a subcategory of $C_\infty A$. Let us derive $C_\infty A$!

$$C_\infty A \xrightarrow[\text{homotopy equivalence}]{\text{quotient by}} K_\infty A \xrightarrow[\text{quasi-isomorphisms}]{\text{localise on}} D_\infty A$$

Nothing is inverted as quasi-isomorphisms have quasi-inverses !

So $D_\infty A = K_\infty A$.

triangulated structure given by

$(\Sigma M)^p := M^{p+1}$; $m_n^{\Sigma M} := (-1)^n m_n^M$

Recall the derived category of Mod-A :

$$\text{Mod-}A \longrightarrow K(\text{Mod-}A) \longrightarrow D(\text{Mod-}A) =: DA$$

Thm. The canonical functor $DA \to D_\infty A$ is an equivalence onto the full subcategory of homologically unital A_∞-modules over A; (1_A acts as id on $H^\bullet(M)$)

For each A_∞-module M over A, M is isomorphic to $H^\bullet(M)$ in $D_\infty A$, where $H^\bullet(M)$ admits a minimal A_∞-module structure.

Implication?

Consider a cochain complex $M^\bullet$ of unital right A-modules. We want to know what additional structure is required to reconstruct $M^\bullet$ from $H^\bullet(M^\bullet)$. The answer is the unique A_∞-module structure on $H^\bullet(M^\bullet)$ over A by the theorem.